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ON THE OPERATIONS OVER INDEX MATRICES

Krassimir T. Atanassov

Department of Bioinformatics and Mathematical Modelling
Institute of Biophysics and Biomedical Engineering
Bilgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 105, Sofia-1113, Bulgaria
E-mail: krat@bas.bg
and
Intelligent Systems Laboratory
Prof. Asen Zlatarov University, Bourgas-8000, Bulgaria

Abstract: In this paper, an extension of the definitions of the operations over index matrices is discussed.
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1 Introduction

The concept of Index Matrix (IM) was introduced in 1984 in [1] and described formally in 1987 in [2]. The basic results, obtained by the author before 2014 were collected in his book [3]. For example, the definition of the IM with elements being real numbers is the following.

Let $\mathcal{I}$ be a fixed set of indices and $\mathcal{R}$ be the set of real numbers. Let operations $\circ, * : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ be fixed. For example, they can be the pairs, $\langle \circ, * \rangle \in \{\langle \times, + \rangle, \langle \text{max}, \text{min} \rangle, \langle \text{min}, \text{max} \rangle\}$, or others.

Let the sets K and L satisfy the condition: $K, L \subset \mathcal{I}$. Let over these sets, the standard set-theoretical operations be defined. We call "IM with real number elements" the object:

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_1 & a_{k_1,l_1} & a_{k_1,l_2} & \dots & a_{k_1,l_n} \\ k_2 & a_{k_2,l_1} & a_{k_2,l_2} & \dots & a_{k_2,l_n} \\ \vdots & & & & \\ k_m & a_{k_m,l_1} & a_{k_m,l_2} & \dots & a_{k_m,l_n} \end{array},$$

where

$$K = \{k_1, k_2, ..., k_m\} \text{ and } L = \{l_1, l_2, ..., l_n\},$$

and for $1 \leq i \leq m$, and for $1 \leq j \leq n : a_{k_i,l_j} \in \mathcal{R}$.

In [3], a lot of operations, relations and operators over index matrices are defined. IMs include as a partial case the standard matrices and four operations over them ("Adition", "Termwise multiplication", "Multiplication" and "Multiplication with a constant") include as partial cases the operations over standard matrices. The rest operations do not have analogues in the theory of standard matrices. It is important to mention that in the IM case, the operations "Adition", "Termwise multiplication" and "Multiplication" are essentially extended compared to the standard case. Each one of them is on two levels: operation on a matrix level and (sub)operation on matrix element level. For example, for the two IMs

$$A = [K, L, \{a_{k_i,l_j}\}]$$

and

$$B = [P, Q, \{b_{p_r,q_s}\}],$$

operation "Multiplication" has the form:

$$A \odot_{(\circ,*)} B = [K \cup (P - L), Q \cup (L - P), \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P - Q \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P - L - K \text{ and } v_w = q_s \in Q \\ \underset{l_j = p_r \in L \cap P}{\circ} a_{k_i,l_j} * b_{p_r,q_s}, & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ 0, & \text{otherwise} \end{cases}$$

Obviously, when $\circ$ is substituted by $+$, the symbol $\underset{j}{\circ}$ is substituted by $\sum_j$.

The idea for operations over two levels was discussed for the first time in 2014 in [3]. There, for the case "otherwise" in the definition of operation "Multiplication" is determined the value "0". Really, this value is one of the possible unitary elements in a group with operation "+" (see, e.g., [4, 5]). But operation "○" can be different than "+". By this reason, in the next section, we extend the definitions of the IM operations.

2 New definitions of index matrix operations

Following [3], we give new definitions of the IM operations, extending the older ones.

Let operations $\circ, * : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be fixed, where X is a fixed set (of natural, real, complex numbers, sentences or predicates, intuitionistic fuzzy pairs, etc.) Let $e_\circ$ and e_* be unitary elements in monoid with support set X. Now, the definitions of the operations have the following forms.

Addition

$$A \oplus_{(\circ)} B = [K \cup P, L \cup Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \\ a_{k_i,l_j} \circ b_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ \\ e_\circ, & \text{otherwise} \end{cases}$$

Of course, if "$\circ$" is substituted by "+", then $a_{k_i,l_j} \circ b_{p_r,q_s} = a_{k_i,l_j} + b_{p_r,q_s}$ and "$e_\circ$" – with "0", while, if "$\circ$" is ".", then $a_{k_i,l_j} \circ b_{p_r,q_s} = a_{k_i,l_j}.b_{p_r,q_s}$ and "$e_\circ$" is substituted by "1".

The definition of

Termwise multiplication

$$A \otimes_{(\circ)} B = [K \cap P, L \cap Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = a_{k_i,l_j} \circ b_{p_r,q_s},$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$ is not changed, while the definition of

Multiplication

$$A \odot_{(\circ,*)} B = [K \cup (P - L), Q \cup (L - P), \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P - Q \\ \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P - L - K \text{ and } v_w = q_s \in Q \\ \\ \underset{l_j = p_r \in L \cap P}{\circ} (a_{k_i,l_j} * b_{p_r,q_s}), & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ \\ e_\circ, & \text{otherwise} \end{cases}$$

changes its unitary element.

The definitions of "Structural subtraction", "Multiplication with a constant" and "Termwise subtraction" do not change.

Obviously, when set X contains as elements sentences or predicates, the (sub)operation on matrix element level can be, e.g., "conjunction (&)", "disjunction ($\vee$)" or another. In the first case, the unitary element "$e_\circ$" will be "$true$" and in the second case – "$false$".

More interesting are the cases, when we have an Intuitionistic Fuzzy IM (IFIM), that has the form

$$[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

$$\equiv \begin{array}{c|ccccc} & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \dots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_i & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle & \dots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle & \dots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \dots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where for every $1 \le i \le m, 1 \le j \le n$: $0 \le \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \le 1$; or an Extended IFIM (EIFIIM)

$$[K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

$$\equiv \begin{array}{l|ccccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots & l_j, \langle \alpha_j^l, \beta_j^l \rangle & \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \dots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle & \dots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle & \dots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \dots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where for every $1 \le i \le m, 1 \le j \le n$:

$$\mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \in [0, 1],$$

$$\alpha_i^k, \beta_i^k, \alpha_i^k + \beta_i^k \in [0, 1],$$

$$\alpha_j^l, \beta_j^l, \alpha_j^l + \beta_j^l \in [0, 1]$$

and

$$K^* = \{\langle k_i, \alpha_i^k, \beta_i^k \rangle | k_i \in K\} = \{\langle k_i, \alpha_i^k, \beta_i^k \rangle | 1 \le i \le m\},$$

$$L^* = \{\langle l_j, \alpha_j^l, \beta_j^l \rangle | l_j \in L\} = \{\langle l_j, \alpha_j^l, \beta_j^l \rangle | 1 \le j \le n\}.$$

In these cases, we use a pair of operations "$(\circ, *)$" that have own unitary elements "$e_\circ$" and "e_*", so that $e_\circ, e_* \in [0, 1]$ and $e_\circ + e_* \in [0, 1]$. Now, for the more extended case of EIFIMs, the above operations have the following forms.

Addition-$(\circ, *)$

$$A \oplus_{(\circ,*)} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$T^* = K^* \cup P^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K \cup P\},$$

$$V^* = L^* \cup Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v\rangle | v_w \in L \cup Q\},$$

$$\alpha_u^t = \begin{cases} \alpha_i^k, & \text{if } t_u \in K - P \\ \alpha_r^p, & \text{if } t_u \in P - K \\ \circ(\alpha_i^k, \alpha_r^p), & \text{if } t_u \in K \cap P \end{cases},$$

$$\beta_w^v = \begin{cases} \beta_j^l, & \text{if } v_w \in L - Q \\ \beta_s^q, & \text{if } t_w \in Q - L \\ * * (\beta_j^l, \beta_s^q), & \text{if } t_w \in L \cap Q \end{cases},$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle, & \text{if } t_u = k_i \in K \\ & \quad \text{and } v_w = l_j \in L - Q \\ & \quad \text{or } t_u = k_i \in K - P \\ & \quad \text{and } v_w = l_j \in L; \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s}\rangle, & \text{if } t_u = p_r \in P \\ & \quad \text{and } v_w = q_s \in Q - L \\ & \quad \text{or } t_u = p_r \in P - K \\ & \quad \text{and } v_w = q_s \in Q; \\ \langle \circ(\mu_{k_i,l_j}, \rho_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ \quad *(\nu_{k_i,l_j}, \sigma_{p_r,q_s})\rangle, & \quad \text{and } v_w = l_j = q_s \in L \cap Q \\ \langle e_\circ, e_*\rangle, & \text{otherwise} \end{cases}$$

Termwise multiplication-$(\circ, *)$

$$A \otimes_{(\circ,*)} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle\}],$$

where

$$T^* = K^* \cap P^* = \{\langle t_u, \alpha_u^t, \beta_u^t\rangle | t_u \in K \cap P\},$$

$$V^* = L^* \cap Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v\rangle | v_w \in L \cap Q\},$$

$$\alpha_u^t = \circ(\alpha_i^k, \alpha_r^p), \text{ for } t_u = k_i = p_r \in K \cap P,$$

$$\beta_w^v = *(\beta_j^l, \beta_s^q), \text{ for } v_w = l_j = q_s \in L \cap Q$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle = \langle \circ(\mu_{k_i,l_j}, \rho_{p_r,q_s}), *(\nu_{k_i,l_j}, \sigma_{p_r,q_s})\rangle.$$

Multiplication-$(\circ, *)$

$$A \odot_{(\circ,*)} B = [T^*, V^*, \langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle\}],$$

where

$$T^* = (K \cup (P - L))^* = \{\langle t_u, \alpha_u^t, \beta_u^t\rangle | t_u \in K \cup (P - L)\},$$

$$V^* = (Q \cup (L - P))^* = \{\langle v_w, \alpha_w^v, \beta_w^v\rangle | v_w \in Q \cup (L - P)\},$$

$$\alpha_u^t = \begin{cases} \alpha_i^k, & \text{if } t_u = k_i \in K \\ \alpha_r^p, & \text{if } t_u = p_r \in P - L - K \end{cases},$$

$$\beta_w^v = \begin{cases} \beta_j^l, & \text{if } v_w = l_j \in L - P - Q \\ \beta_s^q, & \text{if } t_w = q_s \in Q \end{cases},$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle =$$

$$= \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle, & \text{if } t_u = k_i \in K \\ & \quad \text{and } v_w = l_j \in L - P - Q \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s}\rangle, & \text{if } t_u = p_r \in P - L - K \\ & \quad \text{and } v_w = q_s \in Q \\ \Bigg\langle \underset{l_j = p_r \in L \cap P}{\circ} * (\mu_{k_i,l_j}, \rho_{p_r,q_s}), & \text{if } t_u = k_i \in K \\ \underset{l_j = p_r \in L \cap P}{*} \circ (\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \Bigg\rangle, & \quad \text{and } v_w = q_s \in Q \\ \langle e_\circ, e_*\rangle, & \text{otherwise} \end{cases}$$

The so defined operations include as partial cases those from [3]. While writing the present paper, the author saw that in the definitions of these operations in [3], there are some misprints. In the present definitions, they are corrected. For example, in the first definition of operation "addition", in [3] the values for α_u^t and β_w^v are correct, but for the second definition is written that their values are as in the first definition, while in the definition of α_u^t the operation "max" must be changed with "min", and in the definition of β_w^v, the operation "max" must be changed with "min".

The definitions of operations "Addition" and "Termwise multiplication" can be extended. For this aim we must determine two new operations, e.g., $\circ', *' : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ that have own unitary elements "$e_{\circ'}$" and "$e_{*'}$", so that $e_{\circ'}, e_{*'} \in [0, 1]$ and $e_{\circ'} + e_{*'} \in [0, 1]$. Now, we can extend the operations "Addition" and "Termwise multiplication" over EIFIM as follows.

Addition-$(\circ, *; \circ', *')$

$$A \oplus_{(\circ,*)}^{(\circ',*')} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle\}],$$

where

$$T^* = K^* \cup P^* = \{\langle t_u, \alpha_u^t, \beta_u^t\rangle | t_u \in K \cup P\},$$

$$V^* = L^* \cup Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v\rangle | v_w \in L \cup Q\},$$

$$\alpha_u^t = \begin{cases} \alpha_i^k, & \text{if } t_u \in K - P \\ \alpha_r^p, & \text{if } t_u \in P - K \\ \circ'(\alpha_i^k, \alpha_r^p), & \text{if } t_u \in K \cap P \end{cases},$$

$$\beta_w^v = \begin{cases} \beta_j^l, & \text{if } v_w \in L - Q \\ \beta_s^q, & \text{if } t_w \in Q - L \\ **'(\beta_j^l, \beta_s^q), & \text{if } t_w \in L \cap Q \end{cases},$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle, & \text{if } t_u = k_i \in K \\ & \quad \text{and } v_w = l_j \in L - Q \\ & \quad \text{or } t_u = k_i \in K - P \\ & \quad \text{and } v_w = l_j \in L; \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s}\rangle, & \text{if } t_u = p_r \in P \\ & \quad \text{and } v_w = q_s \in Q - L \\ & \quad \text{or } t_u = p_r \in P - K \\ & \quad \text{and } v_w = q_s \in Q; \\ \langle \circ(\mu_{k_i,l_j}, \rho_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ \quad *(\nu_{k_i,l_j}, \sigma_{p_r,q_s})\rangle, & \quad \text{and } v_w = l_j = q_s \in L \cap Q \\ \langle e_\circ, e_*\rangle, & \text{otherwise} \end{cases}$$

Termwise multiplication-$(\circ, *; \circ', *')$

$$A \otimes_{(\circ,*)}^{(\circ',*')} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle\}],$$

where

$$T^* = K^* \cap P^* = \{\langle t_u, \alpha_u^t, \beta_u^t\rangle | t_u \in K \cap P\},$$

$$V^* = L^* \cap Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v\rangle | v_w \in L \cap Q\},$$

$$\alpha_u^t = \circ'(\alpha_i^k, \alpha_r^p), \text{ for } t_u = k_i = p_r \in K \cap P,$$

$$\beta_w^v = *'(\beta_j^l, \beta_s^q), \text{ for } v_w = l_j = q_s \in L \cap Q$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle = \langle \circ(\mu_{k_i,l_j}, \rho_{p_r,q_s}), *(\nu_{k_i,l_j}, \sigma_{p_r,q_s})\rangle.$$

3 Conclusion

In future, some properties of the new operations will be studied and some of their applications will be discussed.

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References

[1] Atanassov K., Conditions in generalized nets, *Proc. of the XIII Spring Conf. of the Union of Bulg. Math.*, Sunny Beach, April 1984, 219-226.

[2] Atanassov K., Generalized index matrices, *Comptes rendus de l'Academie Bulgare des Sciences*, vol.40, 1987, No.11, 15-18.

[3] Atanassov, K., Index Matrices: Towards an Augmented Matrix Calculus, Springer, Cham, 2014.

[4] Cohn, P. Universal Algebra. Harper & Row, New York, 1965.

[5] Kurosh, A. Group Theory. Nauka, Moscow, 1967 (in Russian).