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LEVEL OPERATORS OVER INDEX MATRICES.
PART 2: INDEX MATRICES WITH ELEMENTS FROM A FIXED INTERVAL

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Abstract: For a given index matrix with elements from a fixed interval, level operators are introduced, that modify the elements of the matrix. The basic properties of these operators are studied. An omission of the author’s book [3] has been identified and announced.
Keywords: Index matrix, Level operation.

1 Introduction

The Index Matrices (IMs) and some of their extensions are introduced and discussed in a series of papers and a book (see, e.g., [1, 2, 3]). It is shown that the IMs are extensions of the well-known object “matrix” (see, e.g., [6, 7, 8]). Different operations (only part of them have analogues in standard matrix theory), relations and operators (without analogues in standard matrix theory) are defined over IMs.

In [4], four new operators are introduced over IMs with elements being non-negative real numbers.

In the first (standard) form, the IM is defined as follows (see [1, 2, 3]).

Let $\mathcal{I}$ be a fixed set of indices and φ, ψ be fixed real numbers, so that $\varphi < \psi$.

Let operations $\circ, * : [\varphi, \psi] \times [\varphi, \psi] \to [\varphi, \psi]$ be fixed.

An IM with index sets K and L $(K, L \subset \mathcal{I})$ and elements from interval $[\varphi, \psi]$ is called the object (see, [1, 2, 3]):

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|ccccc} & l_1 & \ldots & l_j & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1} & \ldots & a_{k_1,l_j} & \ldots & a_{k_1,l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & a_{k_i,l_1} & \ldots & a_{k_i,l_j} & \ldots & a_{k_i,l_n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m,l_1} & \ldots & a_{k_m,l_j} & \ldots & a_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, \ldots, k_m\}$, $L = \{l_1, l_2, \ldots, l_n\}$, and for $1 \leq i \leq m$, and $1 \leq j \leq n : a_{k_i,l_j} \in [\varphi, \psi]$.

Let us call these IMs as $[\varphi, \psi]$-IMs, but here we will use only the brief form "IM".

For the IMs

$$A = [K, L, \{a_{k_i,l_j}\}],$$
$$B = [P, Q, \{b_{p_r,q_s}\}],$$

operations that are analogous to the usual matrix operations of addition and termwise multiplication are defined, as well as other, specific ones. Two of them are the following (in the first and the third of the definitions below, the addition from [4] is used):

Addition

$$A \otimes_{(\circ)} B = [K \cup P, L \cup Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ a_{k_i,l_j} \circ b_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ e_\circ, & \text{otherwise} \end{cases},$$

where $e_\circ$ is the unary element in $[\varphi, \psi]$ related to operation $\circ$.

Here and below, if "$\circ$" is "max" or min, then

$$a_{k_i,l_j} \circ b_{p_r,q_s} = \max(a_{k_i,l_j}, b_{p_r,q_s})$$

or

$$a_{k_i,l_j} \circ b_{p_r,q_s} = \min(a_{k_i,l_j}, b_{p_r,q_s}),$$

respectively. When we use operations min, $e_{\text{min}} = \psi$; while for operation max, $e_{\text{max}} = \varphi$.

In [1, 2, 3], for element $e_\circ$ was used 0, but in [4] it was discussed the present sense of $e_\circ$.

Termwise multiplication

$$A \otimes_{(\circ)} B = [K \cap P, L \cap Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = a_{k_i,l_j} \circ b_{p_r,q_s},$$

for $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$.

Multiplication

$$A \odot_{(\circ,*)} B = [K \cup (P - L), Q \cup (L - P), \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P - Q \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P - L - K \text{ and } v_w = q_s \in Q \\ \underset{l_j = p_r \in L \cap P}{\circ} (a_{k_i,l_j} * b_{p_r,q_s}), & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ e_\circ, & \text{otherwise} \end{cases},$$

where $e_\circ$ is the unary element in $[\varphi, \psi]$ related to operation $\circ$.

A lot of relations are defined over two IMs. Here, we use only three of them:

The strict relation "inclusion about value" is

$$A \subset_v B \text{ iff } (K = P) \& (L = Q) \& (\forall k \in K)(\forall l \in L)(a_{k,l} < b_{k,l}).$$

The non-strict relation "inclusion about value" is

$$A \subseteq_v B \text{ iff } (K = P) \& (L = Q) \& (\forall k \in K)(\forall l \in L)(a_{k,l} \le b_{k,l}),$$

Relation "equality about value" is

$$A =_v B \text{ iff } (K = P) \& (L = Q) \& (\forall k \in K)(\forall l \in L)(a_{k,l} = b_{k,l}).$$

In [5], for a first time, level operators were defined over IMs. Here, we continue the research from [5], keeping all notations used there. So, the differences will be more visible.

2 Definition and properties of the new operators

Let us have the IM $A = [K, L, \{a_{k_i,l_j}\}]$, where $a_{k_i,l_j} \in [\varphi, \psi]$. Then we define

$$\mathcal{L}^{>}_{\alpha}(A) = [K, L, \{b_{k_i,l_j}\}],$$

where

$$b_{k_i,l_j} = \begin{cases} a_{k_i,l_j}, & \text{if } a_{k_i,l_j} > \alpha \\ \varphi, & \text{otherwise} \end{cases};$$

$$\mathcal{L}^{\geq}_{\alpha}(A) = [K, L, \{b_{k_i,l_j}\}],$$

where

$$b_{k_i,l_j} = \begin{cases} a_{k_i,l_j}, & \text{if } a_{k_i,l_j} \geq \alpha \\ \varphi, & \text{otherwise} \end{cases}.$$

$$\mathcal{L}^{<}_{\alpha}(A) = [K, L, \{b_{k_i,l_j}\}],$$

where

$$b_{k_i,l_j} = \begin{cases} a_{k_i,l_j}, & \text{if } a_{k_i,l_j} < \alpha \\ \psi, & \text{otherwise} \end{cases};$$

$$\mathcal{L}^{\leq}_{\alpha}(A) = [K, L, \{b_{k_i,l_j}\}],$$

where

$$b_{k_i,l_j} = \begin{cases} a_{k_i,l_j}, & \text{if } a_{k_i,l_j} \leq \alpha \\ \psi, & \text{otherwise} \end{cases}.$$

Let for $K, L \subset \mathcal{I}$:

$$\Phi_{K,L} = [K, L, \{\varphi\}],$$
$$\Psi_{K,L} = [K, L, \{\psi\}].$$

Then for $\alpha \in [\varphi, \psi]$:

$$\mathcal{L}^{>}_{\alpha}(\Phi_{K,L}) = \Phi_{K,L},$$
$$\mathcal{L}^{\geq}_{\alpha}(\Phi_{K,L}) = \Phi_{K,L},$$
$$\mathcal{L}^{<}_{\alpha}(\Psi_{K,L}) = \Psi_{K,L},$$
$$\mathcal{L}^{\leq}_{\alpha}(\Psi_{K,L}) = \Psi_{K,L}.$$

Theorem 1. For each IM A and for every two numbers $\alpha, \beta \in [\varphi, \psi]$:

$$\mathcal{L}^{>}_{\alpha}(\mathcal{L}^{>}_{\beta}(A)) = \mathcal{L}^{>}_{\max(\alpha,\beta)}(A),$$
$$\mathcal{L}^{\geq}_{\alpha}(\mathcal{L}^{\geq}_{\beta}(A)) = \mathcal{L}^{\geq}_{\max(\alpha,\beta)}(A),$$
$$\mathcal{L}^{<}_{\alpha}(\mathcal{L}^{<}_{\beta}(A)) = \mathcal{L}^{<}_{\min(\alpha,\beta)}(A),$$
$$\mathcal{L}^{\leq}_{\alpha}(\mathcal{L}^{\leq}_{\beta}(A)) = \mathcal{L}^{\leq}_{\min(\alpha,\beta)}(A).$$

Proof: Let the above IM A be given. Then

$$B = \mathcal{L}^{>}_{\alpha}(\mathcal{L}^{>}_{\beta}(A)) = \mathcal{L}^{>}_{\alpha}([K, L, \{b_{k_i,l_j}\}]) = [K, L, \{c_{k_i,l_j}\}],$$

where

$$b_{k_i,l_j} = \begin{cases} a_{k_i,l_j}, & \text{if } a_{k_i,l_j} > \beta \\ \varphi, & \text{otherwise} \end{cases}$$

and

$$c_{k_i,l_j} = \begin{cases} b_{k_i,l_j}, & \text{if } b_{k_i,l_j} > \alpha \\ \varphi, & \text{otherwise} \end{cases}.$$

If $a_{k_i,l_j} > \max(\alpha, \beta)$, then

$$c_{k_i,l_j} = b_{k_i,l_j} = a_{k_i,l_j};$$

if $\alpha \geq a_{k_i,l_j} > \beta$, then $b_{k_i,l_j} = a_{k_i,l_j}$, but $c_{k_i,l_j} = \varphi$; if $\beta \geq a_{k_i,l_j} > \alpha$, then $b_{k_i,l_j} = \varphi$ and hence, $c_{k_i,l_j} = \varphi$; if $\min(\alpha, \beta) \geq a_{k_i,l_j}$, then $b_{k_i,l_j} = \varphi$ and hence, $c_{k_i,l_j} = \varphi$.

Therefore,

$$c_{k_i,l_j} = \begin{cases} a_{k_i,l_j}, & \text{if } a_{k_i,l_j} > \max(\alpha, \beta) \\ \\ \varphi, & \text{otherwise} \end{cases},$$

i.e.,

$$B = \mathcal{L}^{>}_{\max(\alpha,\beta)}(A).$$

The next three equalities are proved by the same manner. □

Theorem 2. Let the two IMs A and B be given and let α be an arbitraty real number. Then

(a) $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(\max)} \mathcal{L}^{>}_{\alpha}(B) =_v \mathcal{L}^{>}_{\alpha}(A \oplus_{(\max)} B)$,

(b) $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(\min)} \mathcal{L}^{>}_{\alpha}(B) =_v \mathcal{L}^{>}_{\alpha}(A \oplus_{(\min)} B)$,

(c) $\mathcal{L}^{\geq}_{\alpha}(A) \oplus_{(\max)} \mathcal{L}^{\geq}_{\alpha}(B) =_v \mathcal{L}^{\geq}_{\alpha}(A \oplus_{(\max)} B)$,

(d) $\mathcal{L}^{\geq}_{\alpha}(A) \oplus_{(\min)} \mathcal{L}^{\geq}_{\alpha}(B) =_v \mathcal{L}^{\geq}_{\alpha}(A \oplus_{(\min)} B)$,

(e) $\mathcal{L}^{<}_{\alpha}(A) \oplus_{(\max)} \mathcal{L}^{<}_{\alpha}(B) =_v \mathcal{L}^{<}_{\alpha}(A \oplus_{(\max)} B)$,

(f) $\mathcal{L}^{<}_{\alpha}(A) \oplus_{(\min)} \mathcal{L}^{<}_{\alpha}(B) =_v \mathcal{L}^{<}_{\alpha}(A \oplus_{(\min)} B)$,

(g) $\mathcal{L}^{\leq}_{\alpha}(A) \oplus_{(\max)} \mathcal{L}^{\leq}_{\alpha}(B) =_v \mathcal{L}^{\leq}_{\alpha}(A \oplus_{(\max)} B)$,

(h) $\mathcal{L}^{\leq}_{\alpha}(A) \oplus_{(\min)} \mathcal{L}^{\leq}_{\alpha}(B) =_v \mathcal{L}^{\leq}_{\alpha}(A \oplus_{(\min)} B)$.

Proof: (a) From definition of operation $\oplus_{(\max)}$ we have:

$$\mathcal{L}^{>}_{\alpha}(A \oplus_{(\max)} B)$$

$$= \mathcal{L}^{>}_{\alpha}[K \cup P, L \cup Q, \{c_{t_u,v_w}\}],$$
$$= [K \cup P, L \cup Q, \{d_{t_u,v_w}\}],$$

where

$$d_{t_u,v_w} = \begin{cases} c_{t_u,v_w}, & \text{if } c_{t_u,v_w} > \alpha \\ \\ \varphi, & \text{otherwise} \end{cases}.$$

Sequentially, we will study the different cases for d_{t_u,v_w}.

1. If $c_{t_u,v_w} = a_{k_i,l_j}$, then

$$d_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } a_{k_i,l_j} > \alpha \\ \varphi, & \text{otherwise} \end{cases}$$

and therefore, the elements with indices $\langle k_i, l_j \rangle$ will coincide in the two IMs $\mathcal{L}_{\alpha}^{>}(A \oplus_{(\max)} B)$ and $\mathcal{L}_{\alpha}^{>}(A) \oplus_{(\max)} \mathcal{L}_{\alpha}^{>}(B)$.

2. If $c_{t_u,v_w} = b_{p_r,q_s}$, then

$$d_{t_u,v_w} = \begin{cases} b_{p_r,q_s}, & \text{if } b_{p_r,q_s} > \alpha \\ \varphi, & \text{otherwise} \end{cases}$$

and therefore, the elements with indices $\langle k_i, l_j \rangle$ will coincide in the two IMs $\mathcal{L}_{\alpha}^{>}(A \oplus_{(\max)} B)$ and $\mathcal{L}_{\alpha}^{>}(A) \oplus_{(\max)} \mathcal{L}_{\alpha}^{>}(B)$.

3. If $c_{t_u,v_w} = \max(a_{k_i,l_j}, b_{p_r,q_s})$, then

$$d_{t_u,v_w} = \begin{cases} \max(a_{k_i,l_j}, b_{p_r,q_s}), & \text{if } \max(a_{k_i,l_j}, b_{p_r,q_s}) > \alpha \\ \varphi, & \text{otherwise.} \end{cases}$$

By condition, the elements of A and B are elements of interval $[\varphi, \psi]$. Now, there are the following four subcases:

3.1. if $a_{k_i,l_j} > \alpha$ and $b_{p_r,q_s} > \alpha$, then these elements will keep their values in $\mathcal{L}_{\alpha}^{>}(A)$ and $\mathcal{L}_{\alpha}^{>}(B)$, respectively and their maximum will coincide with the value $\max(a_{k_i,l_j}, b_{p_r,q_s})$ in IM $\mathcal{L}_{\alpha}^{>}(A \oplus_{(\max)} B)$.

3.2 if $a_{k_i,l_j} > \alpha$ and $b_{p_r,q_s} \leq \alpha$, then only the a-element will keep its value in $\mathcal{L}_{\alpha}^{>}(A)$, while the b-element will obtain value φ in $\mathcal{L}_{\alpha}^{>}(B)$. Therefore, in IM $\mathcal{L}_{\alpha}^{>}(A \oplus_{(\max)} B)$the value of d_{t_u,v_w} will be $\max(a_{k_i,l_j}, b_{p_r,q_s}) = a_{k_i,l_j}$. On the other hand side, in IM $\mathcal{L}_{\alpha}^{>}(A) \oplus_{(\max)} \mathcal{L}_{\alpha}^{>}(B)$ its corresponding value will be again $\max(a_{k_i,l_j}, b_{p_r,q_s})$. the value of d_{t_u,v_w}

3.3 if $a_{k_i,l_j} \leq \alpha$ and $b_{p_r,q_s} > \alpha$, then only the b-element will keep its value in $\mathcal{L}_{\alpha}^{>}(B)$, while the a-element will obtain value φ in $\mathcal{L}_{\alpha}^{>}(A)$. Therefore, similarly to case 3.2, we will obtain equality.

3.4 if $a_{k_i,l_j} \leq \alpha$ and $b_{p_r,q_s} \leq \alpha$, then the a- and b-elements will obvain value φ in IMs $\mathcal{L}_{\alpha}^{>}(A)$ and $\mathcal{L}_{\alpha}^{>}(B)$. Therefore, the value of $d_{t_u,v_w} = \max(a_{k_i,l_j}, b_{p_r,q_s})$ in IM $\mathcal{L}_{\alpha}^{>}(A \oplus_{(\max)} B)$ will be equal to φ and to its corresponding value in IM $\mathcal{L}_{\alpha}^{>}(A) \oplus_{(\max)} \mathcal{L}_{\alpha}^{>}(B)$.

4. If $c_{t_u,v_w} = \varphi$, then $d_{t_u,v_w} = \varphi$ and therefore, the elements with indices $\langle k_i, l_j \rangle$ will coincide in the two IMs $\mathcal{L}_{\alpha}^{>}(A \oplus_{(\max)} B)$ and $\mathcal{L}_{\alpha}^{>}(A) \oplus_{(\max)} \mathcal{L}_{\alpha}^{>}(B)$.

Hence, relation $=_v$ between the two IMs exists and when there is at least one pair of a- and b-elements for which Case 3.2 or Case 3.3 is valid, then the relation will be strict $\subset_v$.

(b) – (h) are proved by the same manner. □

The proofs of the next assertions are produced analogously.

Theorem 3. Let the two IMs A and B be given, and $\alpha \in [\varphi, \psi]$. Then

(a) $\mathcal{L}_{\alpha}^{>}(A) \otimes_{(\max)} \mathcal{L}_{\alpha}^{>}(B) =_v \mathcal{L}_{\alpha}^{>}(A \otimes_{(\max)} B)$,

(b) $\mathcal{L}_{\alpha}^{>}(A) \otimes_{(\min)} \mathcal{L}_{\alpha}^{>}(B) =_v \mathcal{L}_{\alpha}^{>}(A \otimes_{(\min)} B)$,

(c) $\mathcal{L}_{\alpha}^{\geq}(A) \otimes_{(\max)} \mathcal{L}_{\alpha}^{\geq}(B) =_v \mathcal{L}_{\alpha}^{\geq}(A \otimes_{(\max)} B)$,

(d) $\mathcal{L}_{\alpha}^{\geq}(A) \otimes_{(\min)} \mathcal{L}_{\alpha}^{\geq}(B) =_v \mathcal{L}_{\alpha}^{\geq}(A \otimes_{(\min)} B)$,

(e) $\mathcal{L}_{\alpha}^{<}(A) \otimes_{(\max)} \mathcal{L}_{\alpha}^{<}(B) =_v \mathcal{L}_{\alpha}^{<}(A \otimes_{(\max)} B)$,

(f) $\mathcal{L}_{\alpha}^{<}(A) \otimes_{(\min)} \mathcal{L}_{\alpha}^{<}(B) =_v \mathcal{L}_{\alpha}^{<}(A \otimes_{(\min)} B)$,

(g) $\mathcal{L}_{\alpha}^{\leq}(A) \otimes_{(\max)} \mathcal{L}_{\alpha}^{\leq}(B) =_v \mathcal{L}_{\alpha}^{\leq}(A \otimes_{(\max)} B)$,

(h) $\mathcal{L}_{\alpha}^{\leq}(A) \otimes_{(\min)} \mathcal{L}_{\alpha}^{\leq}(B) =_v \mathcal{L}_{\alpha}^{\leq}(A \otimes_{(\min)} B)$.

Theorem 4. Let the two IMs A and B be given. Then

(a) $\mathcal{L}_{\alpha}^{>}(A \odot_{(\max,\min)} B) =_v \mathcal{L}_{\alpha}^{>}(A) \odot_{(\max,\min)} \mathcal{L}_{\alpha}^{>}(B)$,

(b) $\mathcal{L}_{\alpha}^{>}(A \odot_{(\min,\max)} B) =_v \mathcal{L}_{\alpha}^{>}(A) \odot_{(\min,\max)} \mathcal{L}_{\alpha}^{>}(B)$,

(c) $\mathcal{L}_{\alpha}^{\geq}(A \odot_{(\max,\min)} B) =_v \mathcal{L}_{\alpha}^{\geq}(A) \odot_{(\max,\min)} \mathcal{L}_{\alpha}^{\geq}(B)$,

(d) $\mathcal{L}_{\alpha}^{\geq}(A \odot_{(\min,\max)} B) =_v \mathcal{L}_{\alpha}^{\geq}(A) \odot_{(\min,\max)} \mathcal{L}_{\alpha}^{\geq}(B)$,

(e) $\mathcal{L}_{\alpha}^{<}(A \odot_{(\max,\min)} B) =_v \mathcal{L}_{\alpha}^{<}(A) \odot_{(\max,\min)} \mathcal{L}_{\alpha}^{<}(B)$,

(f) $\mathcal{L}_{\alpha}^{<}(A \odot_{(\min,\max)} B) =_v \mathcal{L}_{\alpha}^{<}(A) \odot_{(\min,\max)} \mathcal{L}_{\alpha}^{<}(B)$,

(g) $\mathcal{L}_{\alpha}^{\leq}(A \odot_{(\max,\min)} B) =_v \mathcal{L}_{\alpha}^{\leq}(A) \odot_{(\max,\min)} \mathcal{L}_{\alpha}^{\leq}(B)$,

(h) $\mathcal{L}_{\alpha}^{\leq}(A \odot_{(\min,\max)} B) =_v \mathcal{L}_{\alpha}^{\leq}(A) \odot_{(\min,\max)} \mathcal{L}_{\alpha}^{\leq}(B)$.

Proof: (h) From definition of operation $\odot_{(\min)}$ we have:

$$\begin{aligned} &\mathcal{L}_{\alpha}^{\leq}(A \odot_{(\min,\max)} B) \\ &= \mathcal{L}_{\alpha}^{\leq}[K \cup (P - L), Q \cup (L - P), \{c_{t_u,v_w}\}], \\ &= [K \cup (P - L), Q \cup (L - P), \{d_{t_u,v_w}\}], \end{aligned}$$

where

$$d_{t_u,v_w} = \begin{cases} c_{t_u,v_w}, & \text{if } c_{t_u,v_w} \leq \alpha \\ \psi, & \text{otherwise} \end{cases}.$$

Sequentially, we will study the different cases for d_{t_u,v_w}.

1. If $c_{t_u,v_w} = a_{k_i,l_j}$, then

$$d_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } a_{k_i,l_j} \leq \alpha \\ \psi, & \text{otherwise} \end{cases}$$

and therefore, the elements with indices $\langle k_i, l_j \rangle$ will coincide in the two IMs $\mathcal{L}^{>}_{\alpha}(A \oplus_{(\min,\max)} B)$ and $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(\min,\max)} \mathcal{L}^{>}_{\alpha}(B)$.

2. If $c_{t_u,v_w} = b_{p_r,q_s}$, then

$$d_{t_u,v_w} = \begin{cases} b_{p_r,q_s}, & \text{if } b_{p_r,q_s} \leq \alpha \\ \psi, & \text{otherwise} \end{cases}$$

and therefore, the elements with indices $\langle p_r, q_s \rangle$ will coincide in the two IMs $\mathcal{L}^{>}_{\alpha}(A \oplus_{(\min,\max)} B)$ and $\mathcal{L}^{>}_{\alpha}(A) \oplus_{(\min,\max)} \mathcal{L}^{>}_{\alpha}(B)$.

3. If $c_{t_u,v_w} = \min\limits_{l_j = p_r \in L \cap P} \max(a_{k_i,l_j}, b_{p_r,q_s})$, then

$$d_{t_u,v_w} = \begin{cases} \min\limits_{l_j = p_r \in L \cap P} \max(a_{k_i,l_j}, b_{p_r,q_s}), & \text{if } \min\limits_{l_j = p_r \in L \cap P} \max(a_{k_i,l_j}, b_{p_r,q_s}) \leq \alpha \\ \psi, & \text{otherwise.} \end{cases}$$

By condition, the elements of A and B are elements of interval $[\varphi, \psi]$. Now, there are the following three subcases.

3.1. if for every indices $\langle k_i, l_j \rangle = \langle p_r, q_s \rangle$: $\max(a_{k_i,l_j}, b_{p_r,q_s}) \leq \alpha$, then $a_{k_i,l_j} \leq \alpha$ and $b_{p_r,q_s} \leq \alpha$ and hence these elements will keep their values in $\mathcal{L}^{\leq}_{\alpha}(A)$ and $\mathcal{L}^{\leq}_{\alpha}(B)$, respectively and their minimum will coincide with the value $\min\limits_{l_j = p_r \in L \cap P} \max(a_{k_i,l_j}, b_{p_r,q_s})$ in IM $\mathcal{L}^{\leq}_{\alpha}(A \oplus_{(\min,\max)} B)$.

3.2. if there are indices $\langle k_i, l_j \rangle = \langle p_r, q_s \rangle$ such that $\max(a_{k_i,l_j}, b_{p_r,q_s}) \leq \alpha$ and other for which $\max(a_{k_i,l_j}, b_{p_r,q_s}) > \alpha$, then only the first type of elements will keep their values in $\mathcal{L}^{\leq}_{\alpha}(A)$ and $\mathcal{L}^{\leq}_{\alpha}(B)$, respectively, while at least one of the components of each pair of the second type of elements will obtain value ψ in its IM. But the value of d_{t_u,v_w} in IM $\mathcal{L}^{\leq}_{\alpha}(A \oplus_{(\min,\max)} B)$ will be $\min\limits_{l_j = p_r \in L \cap P} \max(a_{k_i,l_j}, b_{p_r,q_s})$ and it will be the same in IM $\mathcal{L}^{\leq}_{\alpha}(A) \oplus_{(\min,\max)} \mathcal{L}^{\leq}_{\alpha}(B)$ because existing of elements a_{k_i,l_j}, b_{p_r,q_s} from the first type in the minimum that will participate in min-expression together ψ.

3.3. if for every indices $\langle k_i, l_j \rangle = \langle p_r, q_s \rangle$: $\min\limits_{l_j = p_r \in L \cap P} \max(a_{k_i,l_j}, b_{p_r,q_s}) > \alpha$, then at least one of the respective elements in IMs $\mathcal{L}^{\leq}_{\alpha}(A)$ and $\mathcal{L}^{\leq}_{\alpha}(B)$ will obvain value ψ. Therefore, the value of d_{t_u,v_w} in IM $\mathcal{L}^{\leq}_{\alpha}(A \oplus_{(\min,\max)} B)$ will be equal to ψ, as well as its corresponding value in IM $\mathcal{L}^{\leq}_{\alpha}(A) \oplus_{(\min,\max)} \mathcal{L}^{\leq}_{\alpha}(B)$.

4. If $c_{t_u,v_w} = \psi$, then $d_{t_u,v_w} = \psi$ and therefore, the elements with indices $\langle k_i, l_j \rangle$ will coincide in the two IMs $\mathcal{L}^{\leq}_{\alpha}(A \oplus_{(\min,\max)} B)$ and $\mathcal{L}^{\leq}_{\alpha}(A) \oplus_{(\min,\max)} \mathcal{L}^{\leq}_{\alpha}(B)$.

Hence, relation $=_v$ between the two IMs exists.

(a) – (g) are proved by the same manner. □

3 On an error in [3]

Writing the present paper, the author found an error in his book [3] that will be corrected here.

Let $\mathcal{I}$ be a fixed set and

$$K = \{k_1, k_2, ..., k_m\} \subseteq \mathcal{I},$$

$$L = \{l_1, l_2, ..., l_n\} \subseteq \mathcal{I}.$$

The Extended Intuitionistic Fuzzy Index Matrices (EIFIM) is defined by:

$$[K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

$$\equiv \begin{array}{c|ccccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots & l_j, \langle \alpha_j^l, \beta_j^l \rangle & \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \dots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle & \dots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle & \dots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \dots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$:

$$\mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \in [0, 1],$$

$$\alpha_i^k, \beta_i^k, \alpha_i^k + \beta_i^k \in [0, 1],$$

$$\alpha_j^l, \beta_j^l, \alpha_j^l + \beta_j^l \in [0, 1]$$

and here and below,

$$K^* = \{\langle k_i, \alpha_i^k, \beta_i^k \rangle | k_i \in K\} = \{\langle k_i, \alpha_i^k, \beta_i^k \rangle | 1 \leq i \leq m\},$$

$$L^* = \{\langle l_j, \alpha_j^l, \beta_j^l \rangle | l_j \in L\} = \{\langle l_j, \alpha_j^l, \beta_j^l \rangle | 1 \leq j \leq n\}.$$

Therefore,

$$K^* \subseteq K \times [0, 1] \times [0, 1],$$

$$L^* \subseteq L \times [0, 1] \times [0, 1].$$

Let for $P, Q \subset \mathcal{I}$:

$$P^* = \{\langle p_r, \alpha_r^p, \beta_r^p \rangle | p_r \in P\},$$

$$Q^* = \{\langle q_s, \alpha_s^q, \beta_s^q \rangle | q_s \in Q\}.$$

Then

$$K^* \subset P^* \text{ iff } (K \subset P) \ \& \ (\forall k_i = p_i \in K)((\alpha_i^k < \alpha_i^p) \ \& \ (\beta_i^k > \beta_i^p)).$$

$$K^* \subseteq P^* \text{ iff } (K \subseteq P) \ \& \ (\forall k_i = p_i \in K)((\alpha_i^k \leq \alpha_i^p) \ \& \ (\beta_i^k \geq \beta_i^p)).$$

For the EIFIMs $A = [K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}], B = [P^*, Q^*, \{\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle\}]$, operations that are analogous to the usual matrix operations of addition and multiplication are defined, as well as other specific ones (see [3, 5]).

Addition-$(\circ)$

$$A \oplus_{(\circ)} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle\}],$$

where

$$T^* = K^* \cup P^* = \{\langle t_u, \alpha_u^t, \beta_u^t\rangle | t_u \in K \cup P\},$$

$$V^* = L^* \cup Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v\rangle | v_w \in L \cup Q\},$$

$$\alpha_u^t = \begin{cases} \alpha_i^k, & \text{if } t_u \in K - P \\ \\ \alpha_r^p, & \text{if } t_u \in P - K \\ \\ \circ_1(\alpha_i^k, \alpha_r^p), & \text{if } t_u = k_i = p_r \in K \cap P \end{cases},$$

$$\beta_u^t = \begin{cases} \beta_i^k, & \text{if } t_u = k_i \in K - P \\ \\ \beta_r^p, & \text{if } t_u = p_r \in P - K \\ \\ \circ_2(\beta_i^k, \beta_r^p & \text{if } t_u = k_i = p_r \in K \cap P \end{cases},$$

$$\alpha_w^v = \begin{cases} \alpha_j^l, & \text{if } v_w = l_j \in L - Q \\ \\ \alpha_s^q, & \text{if } v_u = q_s \in Q - L \\ \\ \circ_1(\alpha_i^k, \alpha_r^p), & \text{if } v_w = l_j = q_s \in L \cap Q \end{cases},$$

$$\beta_w^v = \begin{cases} \beta_j^l, & \text{if } v_w = l_j \in L - Q \\ \\ \beta_s^q, & \text{if } v_w = q_s \in Q - L \\ \\ \circ_2(\beta_j^l, \beta_s^q), & \text{if } v_w = l_j = q_s \in L \cap Q \end{cases},$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w}\rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle, & \text{if } t_u = k_i \in K \\ & \quad \text{and } v_w = l_j \in L - Q \\ & \quad \text{or } t_u = k_i \in K - P \\ & \quad \text{and } v_w = l_j \in L; \\ \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s}\rangle, & \text{if } t_u = p_r \in P \\ & \quad \text{and } v_w = q_s \in Q - L \\ & \quad \text{or } t_u = p_r \in P - K \\ & \quad \text{and } v_w = q_s \in Q; \\ \\ \langle \circ_1(\mu_{k_i,l_j}, \rho_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ \quad \circ_2(\nu_{k_i,l_j}, \sigma_{p_r,q_s})\rangle, & \quad \text{and } v_w = l_j = q_s \in L \cap Q \\ \\ \langle 0, 1\rangle, & \text{otherwise} \end{cases}$$

Termwise multiplication-($\circ$**)**

$$A \otimes_{(\circ)} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$T^* = K^* \cap P^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K \cap P\},$$
$$V^* = L^* \cap Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v \rangle | v_w \in L \cap Q\},$$

for $t_u = k_i = p_r \in K \cap P$:

$$\alpha_u^t = \min(\alpha_i^k, \alpha_r^p),$$
$$\beta_u^t = \max(\beta_i^k, \beta_r^p),$$

for $v_w = l_j = q_s \in L \cap Q$:

$$\alpha_w^v = \min(\alpha_j^l, \alpha_s^q),$$
$$\beta_w^v = \max(\beta_j^l, \beta_s^q),$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \langle \circ_1(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \circ_2(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle.$$

Let "$*$" be another operation, e.g., among these from Section 2, and let it have sub-operations $\langle *_1, *_2 \rangle$.

Multiplication-($\circ, *$**)**

$$A \odot_{(\circ,*)} B = [T^*, V^*, \langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle \}],$$

where

$$T^* = (K \cup (P - L))^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K \cup (P - L)\},$$
$$V^* = (Q \cup (L - P))^* = \{\langle v_w, \alpha_w^v, \beta_w^v \rangle | v_w \in Q \cup (L - P)\},$$

$$\alpha_u^t = \begin{cases} \alpha_i^k, & \text{if } t_u = k_i \in K \\ \alpha_r^p, & \text{if } t_u = p_r \in P - L \end{cases},$$

$$\beta_u^t = \begin{cases} \beta_i^k, & \text{if } t_u = k_i \in K \\ \beta_r^p, & \text{if } t_u = p_r \in P - L \end{cases},$$

$$\alpha_w^v = \begin{cases} \alpha_j^l, & \text{if } v_w = l_j \in L - P \\ \alpha_s^q, & \text{if } t_w = q_s \in Q \end{cases},$$

$$\beta_w^v = \begin{cases} \beta_j^l, & \text{if } v_w = l_j \in L - P \\ \beta_s^q, & \text{if } t_w = q_s \in Q \end{cases},$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle =$$

$$= \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \\ & \quad \text{and } v_w = l_j \in L - P - Q \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P - L - K \\ & \quad \text{and } v_w = q_s \in Q \\ \langle \underset{l_j = p_r \in L \cap P}{\circ_1} (*_1(\mu_{k_i,l_j}, \rho_{p_r,q_s})), & \text{if } t_u = k_i \in K \\ \quad \underset{l_j = p_r \in L \cap P}{\circ_2} (*_2(\nu_{k_i,l_j}, \sigma_{p_r,q_s})) \rangle, & \text{and } v_w = q_s \in Q \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

In [3], the definitions of β_u^t and α_w^v are omitted for the three operations.

4 Conclusion

In a next research the case, when the level operators are applied over Intuitionistic Fuzzy IMs (IFIMs) and Extended IFIMs (EIFIMs) will be discussed. Other types of level operators will be introduced.

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