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ON 3-DIMENSIONAL MULTILAYER MATRICES AND OPERATIONS WITH THEM

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Abstract: In this paper will be introduced definition of a 3-dimensional multilayer extended index matrix. Also will be expanded aggregation operations related to a 3-dimensional extended index matrix and some properties of them will be researched.
Keywords: Multilayer index matrix, Operation, Aggregation.

1 Introduction

The index matrices (IM) are introduced in 1984 in [1, 2] and summirized by Atanassov in his book [3]. Apparatus extended index matrices (EIMs) is introduced in [5, 3, 10].

For the needs of the present research we will remain the definition of the 3-dimensional extended index matrix (3D-EIM) and some operations over it in section 2. In this section also be defined 3-dimensional multilayer extended index matrix (3D-MLEIM). In sections 3 will be summarized and expanded the aggregation operations, defined in [3, 7, 9]. In section 4, some their properties will be discussed.

2 Short remarks to a 3D-Extended index matrix

Let us started with the definition of the 3D-extended index matrix from [3] and its extension in [10].

2.1 Definition of a 3D-Extended index matrix

The Intuitionistic Fuzzy Pair (IFP) [4, 6] is an object with the form $\langle a, b\rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$, that is used as an evaluation of some object or process. Its components (a and b)

are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degree of correctness and non-correctness, etc. Let $\mathcal{I}$ be a fixed set of indexes,

$$\mathcal{I}^n = \{\langle i_1, i_2, ..., i_n\rangle | (\forall j : 1 \le j \le n)(i_j \in \mathcal{I})\}$$

$$\text{and } \mathcal{I}^* = \bigcup_{1\le n\le\infty} I^n.$$

Let $\mathcal{X}$ be a fixed set of some objects. In the particular cases, they can be either real numbers, or only the numbers 0 or 1, or logical variables, propositions or predicates, IFPs, function etc.

A "3D-Extended Index Matrix" (3D-EIM) with index sets K, L and H $(K, L, H \subset \mathcal{I}^*)$ and elements from set $\mathcal{X}$ we denoted the object:

$$[K, L, H, \{a_{k_i,l_j,h_g}\}] = \left\{ \begin{array}{c|ccccc} h_g & l_1 & \dots & l_j & \dots & l_n \\ \hline k_1 & a_{k_1,l_1,h_g} & \cdots & a_{k_1,l_j,h_g} & \cdots & a_{k_1,l_n,h_g} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_i & a_{k_i,l_1,h_g} & \cdots & a_{k_i,l_j,h_g} & \cdots & a_{k_i,l_n,h_g} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_m & a_{k_m,l_1,h_g} & \cdots & a_{k_m,l_j,h_g} & \cdots & a_{k_m,l_n,h_g} \end{array} \;\middle|\; h_g \in H \right\}$$

where $K = \{k_1, k_2, \dots, k_m\}$, $L = \{l_1, l_2, \dots, l_n\}$, $H = \{h_1, h_2, \dots, h_f\}$, and for $1 \le i \le m,\ 1 \le j \le n,\ 1 \le g \le f : a_{k_i,l_j,h_g} \in \mathcal{X}$.

Following [3, 10], let $3D{-}EIM_{\mathcal{R}}$ be the set of all 3D-EIMs with elements being real numbers; $3D - EIM_{\{0,1\}}$ be the set of all $(0,1)$-3D-EIMs with elements being 0 or 1; $3D - EIM_{\mathcal{P}}$ be the set of all 3D-EIMs with elements – predicates; $3D - EIM_{\mathcal{IFP}}$ be the set of all 3D-EIMs with elements – IFPs and $3D - EIM_{\mathcal{F}E}$ – the set of all 3D-EIMs with elements – 1-argument functions $\in \mathcal{F}^1$.

2.2 Definition of 3D-Multilayer Extended index matrix (3D-MLEIM)

Let us begin section with a definition of $3D$-multilayer extended index matrix A (3D-MLEIM) with P-levels (layers) of use of a dimension K, Q-levels(layers) of use of a dimension L and R-levels(layers) of use of a dimension H as follows:

$$A = [K, L, H, \{a_{K^{(p)}_{i,d}, L^{(q)}_{j,b}, H^{(r)}_{g,c}}\}]$$

$$= \left\{ \begin{array}{c|ccccc} H^{(R)}_g \in H & L^{(Q)}_1 & \cdots & L^{(Q)}_j & \cdots & L^{(Q)}_n \\ \hline K^{(P)}_1 & a_{K^{(P)}_1, L^{(Q)}_1, H^{(R)}_g} & \cdots & a_{K^{(P)}_1, L^{(Q)}_j, H^{(R)}_g} & \cdots & a_{K^{(P)}_1, L^{(Q)}_n, H^{(R)}_g} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ K^{(P)}_i & a_{K^{(P)}_i, L^{(Q)}_1, H^{(R)}_g} & \cdots & a_{K^{(P)}_i, L^{(Q)}_j, H^{(R)}_g} & \cdots & a_{K^{(P)}_i, L^{(Q)}_n, H^{(R)}_g} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ K^{(P)}_m & a_{K^{(P)}_m, L^{(Q)}_1, H^{(R)}_g} & \cdots & a_{K^{(P)}_m, L^{(Q)}_j, H^{(R)}_g} & \cdots & a_{K^{(P)}_m, L^{(Q)}_n, H^{(R)}_g} \end{array} \right\}$$

where

$$K=\{K_1^{(P)},K_2^{(P)},\dots,K_i^{(P)},\dots,K_m^{(P)}\},$$

$$K_i^{(P)}=\left\{K_{i,1}^{(P-1)},K_{i,2}^{(P-1)},\dots,K_{i,x}^{(P-1)},\dots,K_{i,I}^{(P-1)}\right\}\text{ for } 1\le i\le m$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$K_u^{(1)}=\left\{K_{u,1}^{(0)},K_{u,2}^{(0)},\dots,K_{u,U}^{(0)}\right\}$$

i.e. $p-$th layer of dimension K of the multilayer matrix, where $(1\le p\le p)$, is performed by

$$K_{u_*}^{(p)}=\left\{K_{u_{*,1}}^{(p-1)},K_{u_{*,2}}^{(p-1)},\dots,K_{u_{*,U_*}}^{(p-1)}\right\}\text{ for } 1\le p\le P$$

$$L=\{L_1^{(Q)},L_2^{(Q)},\dots,L_j^{(Q)},\dots,L_n^{(Q)}\},$$

$$L_j^{(Q)}=\{L_{j,1}^{(Q-1)},L_{j,2}^{(Q-1)},\dots,L_{j,y}^{(Q-1)},\dots,L_{j,J}^{(Q-1)}\}\text{ for } 1\le j\le n$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$L_v^{(1)}=\left\{L_{v,1}^{(0)},L_{v,2}^{(0)},\dots,l_{v,V}^{(0)}\right\}$$

i.e. $q-$th layer of dimension Q of the multilayer matrix is performed by

$$L_{v_*}^{(q)}=\left\{L_{v_{*,1}^{(q-1)}},L_{v_{*,2}}^{(q-1)},\dots,L_{v_{*,V_*}}^{(q-1)}\right\}\text{ for } 1\le q\le Q$$

$$H=\{H_1^{(R)},H_2^{(R)},\dots,H_g^{(R)},\dots,H_f^{(R)}\},$$

$$H_g^{(R)}=\{H_{g,1}^{(R-1)},H_{g,2}^{(R-1)},\dots,H_{g,z}^{(R-1)},\dots,H_{g,G}^{(R-1)}\}\text{ for } 1\le g\le f$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$H_w^{(1)}=\left\{H_{w,1}^{(0)},H_{w,2}^{(0)},\dots,H_{w,W}^{(0)}\right\}$$

i.e. $r-$th layer of dimension H of the multilayer matrix is performed by

$$H_{w_*}^{(r)}=\left\{H_{w_{*,1}}^{(r-1)},H_{w_{*,2}}^{(r-1)},\dots,H_{w_{*,W_*}}^{(r-1)}\right\}\text{ for } 1\le r\le R$$

and $(K,L,H\subset I^*)$, and for $1\le i\le I,\ 1\le j\le J,\ 1\le g\le G,\ 1\le p\le P,\ 1\le q\le Q,\ 1\le r\le R,\ 1\le d\le I,\ 1\le b\le J,\ 1\le c\le G: a_{K_{i,d}^{(p)},L_{j,b}^{(q)},H_{g,c}^{(r)}}\in\mathcal{X}$,

2.3 Operations over 3-dimensional extended index matrix

2.3.1 Aggregation operations over 3D-Extended index matrix

Let 3D-EIM

$$A = [K, L, H, \{a_{k_i,l_j,h_g}\}], (K, L, H \subset \mathcal{I}^*)$$

be given and let for $1 \le i \le m, 1 \le j \le n, 1 \le g \le f$, $k_0 \notin K, l_0 \notin L, h_0 \notin H$ and $a_{k_i,l_j,h_g} \in \mathcal{X}$. Let $\circ : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X}$ and $* : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X}$ and

$$\circ \in \begin{cases} \{``+", ``\times", ``average", ``max", ``min"\}, & \text{if } A \in 3D-EIM_{\mathcal{R}} \\ & \text{or } A \in 3D-EIM_{\mathcal{FE}}; \\ \{``max", ``min"\}, & \text{if } A \in 3D-EIM_{\{0,1\}} \\ \{``\wedge", ``\vee"\}, & \text{if } A \in 3D-EIM_{\mathcal{P}} \\ & \text{or } A \in 3D-EIM_{\mathcal{I}FP} \end{cases}$$

In the case of $3D - EIM_{\mathcal{I}FP}$, in aggregation operations can participate aggregating pair operations $(\circ, *)$, which elements are applied respectively on the first and second element of IFP, where

$$\langle \circ, * \rangle \in \{\langle min, min\rangle, \langle min, max\rangle, \langle max, min\rangle, \langle min, average\rangle,$$

$$\langle average, average\rangle, \langle average, min\rangle\}.$$

In all other cases, we use only one operation $(\circ)$.

The aggregation operations over the matrix $A(3D - EIM)$ have the forms, as follows [9]:

$(\circ) - \alpha_K$**-aggregation**

$$\alpha_{(K,\circ)}(A, k_0)$$

$$= \left\{ \begin{array}{c|cccc} h_g & l_1 & l_2 & \cdots & l_n \\ \hline k_0 & \underset{1\le i\le m}{\circ} a_{k_i,l_1,h_g} & \underset{1\le i\le m}{\circ} a_{k_i,l_2,h_g} & \cdots & \underset{1\le i\le m}{\circ} a_{k_i,l_n,h_g} \end{array} | h_g \in H \right\}$$

$(\circ) - \alpha_L$**-aggregation**

$$\alpha_{(L,\circ)}(A, l_0) = \left\{ \begin{array}{c|c} h_g & l_0 \\ \hline k_1 & \underset{1\le j\le n}{\circ} a_{k_1,l_j,h_g} \\ k_2 & \underset{1\le j\le n}{\circ} a_{k_2,l_j,h_g} \\ \vdots & \vdots \\ k_m & \underset{1\le j\le n}{\circ} a_{k_m,l_j,h_g} \end{array} \mid h_g \in H \right\}$$

$(\circ) - \alpha_H$**-aggregation**

$$\alpha_{(H,\circ)}(A, h_0) = \left\{ \begin{array}{c|c} l_j & h_0 \\ \hline k_1 & \underset{1\le g\le f}{\circ} a_{k_1,l_1,h_g} \\ k_2 & \underset{1\le g\le f}{\circ} a_{k_2,l_2,h_g} \\ \vdots & \vdots \\ k_m & \underset{1\le g\le f}{\circ} a_{k_m,l_n,h_g} \end{array} \mid l_j \in L \right\}$$

$(\circ) - \alpha_{(K,L)}$**-aggregation**

$$\alpha_{(K,L,\circ)}(A, \langle k_0, l_0\rangle) = \begin{array}{c|cccc} & h_1 & h_2 & \cdots & h_f \\ \hline \langle k_0, l_0\rangle & \underset{\substack{1\le i\le m \\ 1\le j\le n}}{\circ} a_{k_i,l_j,h_1} & \underset{\substack{1\le i\le m \\ 1\le j\le n}}{\circ} a_{k_i,l_j,h_2} & \cdots & \underset{\substack{1\le i\le m \\ 1\le j\le n}}{\circ} a_{k_i,l_j,h_f} \end{array};$$

$(\circ) - \alpha_{(K,H)}$**-aggregation**

$$\alpha_{(K,H,\circ)}(A, \langle k_0, h_0\rangle) = \begin{array}{c|cccc} & l_1 & l_2 & \cdots & l_n \\ \hline \langle k_0, h_0\rangle & \underset{\substack{1\le i\le m \\ 1\le g\le f}}{\circ} a_{k_i,l_1,h_g} & \underset{\substack{1\le i\le m \\ 1\le g\le f}}{\circ} a_{k_i,l_2,h_g} & \cdots & \underset{\substack{1\le i\le m \\ 1\le g\le f}}{\circ} a_{k_i,l_n,h_g} \end{array};$$

$(\circ) - \alpha_{(L,H)}$**-aggregation**

$$\alpha_{(L,H,\circ)}(A, \langle l_0, h_0\rangle) = \begin{array}{c|cccc} & k_1 & k_2 & \cdots & k_m \\ \hline \langle l_0, h_0\rangle & \underset{\substack{1\le j\le n \\ 1\le g\le f}}{\circ} a_{k_1,l_j,h_g} & \underset{\substack{1\le j\le n \\ 1\le g\le f}}{\circ} a_{k_2,l_j,h_g} & \cdots & \underset{\substack{1\le j\le n \\ 1\le g\le f}}{\circ} a_{k_m,l_j,h_g} \end{array}.$$

2.3.2 Transposition

For every three index sets K, L and H exist 6-1 (= 3!-1) transposed EIMs of a given matrix A as follows [3]:

$$\left[K, L, H, \{a_{k_i,l_j,h_g}\}\right]^{[1,3,2]} = \left[K, H, L, \{a_{k_i,h_g,l_j}\}\right]$$
$$\left[K, L, H, \{a_{k_i,l_j,h_g}\}\right]^{[2,1,3]} = \left[L, K, H\{a_{l_j,k_i,h_g}\}\right]$$
$$\left[K, L, H, \{a_{k_i,l_j,h_g}\}\right]^{[2,3,1]} = \left[L, H, K, \{a_{l_j,h_g,k_i}\}\right]$$
$$\left[K, L, H, \{a_{k_i,l_j,h_g}\}\right]^{[3,1,2]} = \left[H, K, L, \{a_{h_g,k_i,l_j}\}\right]$$
$$\left[K, L, H, \{a_{k_i,l_j,h_g}\}\right]^{[3,2,1]} = \left[H, L, K, \{a_{h_g,l_j,k_i}\}\right].$$

2.3.3 Projection

Let be given 3D-EIM $A = [K, L, H, \{a_{k_i,l_j,h_g}\}]$ and $M \subseteq K$, $N \subseteq L$ and $U \subseteq H$, then

$$pr_{M,N,U}A = [M, N, U, \{b_{k_i,l_j,h_g}\}],$$

where for each $k_i \in M$, $l_j \in N$ and $h_g \in U$, $b_{k_i,l_j,h_g} = a_{k_i,l_j,h_g}$.

3 Generalized aggregation operations

3.1 Generalized aggregation operations of 3-dimensional extended index matrix

Let $3D - EIM$ A be given, that

$$[K, L, H, \{a_{k_{i,d},l_{j,b},h_{g,c}}\}]$$

$$= \left\{ \begin{array}{c|ccccccc} H_g \in H & L_1 & \cdots & l_{j,1} & \cdots & l_{j,J} & \cdots & L_n \\ \hline K_1 & a_{K_1,L_1,H_g} & \cdots & a_{K_1,l_{j_1},H_g} & \cdots & a_{K_1,l_{j_J},H_g} & \cdots & a_{K_1,L_n,H_g} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \\ k_{i,1} & a_{k_{i,1},L_1,H_g} & \cdots & a_{k_{,1},l_{j,1},H_g} & \cdots & a_{k_{i,1},l_{j,J},H_g} & \cdots & a_{k_{i,1},L_n,H_g} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{i,I} & a_{k_{i,I},L_1,H_g} & \cdots & a_{k_{i,I},l_{j,1},H_g} & \cdots & a_{k_{i,I},l_{j,J},H_g} & \cdots & a_{k_{i,I},L_n,H_g} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ K_m & a_{K_m,L_1,H_g} & \cdots & a_{K_m,l_{j_1},H_g} & \cdots & a_{K_m,l_{j_J},H_g} & \cdots & a_{K_m,L_n,H_g} \end{array} \right\}$$

where

$$K = \{K_1, K_2, \ldots, K_i, \ldots, K_m\}, K_i = \{k_{i,1}, k_{i,2}, \ldots, k_{i,I}\} \text{ for } 1 \leq i \leq m,$$

$$L = \{L_1, L_2, \ldots, L_j, \ldots, L_n\}, L_j = \{l_{j,1}, l_{j,2}, \ldots, l_{j,J}\} \text{ for } 1 \leq j \leq n,$$

$$H = \{H_1, H_2, \ldots, H_g, \ldots, H_f\}, H_g = \{h_{g,1}, h_{g,2}, \ldots, h_{g,G}\} \text{ for } 1 \leq g \leq f$$

and $(K, L, H \subset I^*)$, and for $1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq g \leq f,\ 1 \leq d \leq I,\ 1 \leq b \leq J,\ 1 \leq c \leq G : a_{k_{i,d},l_{j,b},h_{g,c}} \in \mathcal{X}$.

Let $k_{i,0} \notin K, l_{j,0} \notin L, h_{g,0} \notin H$. Let us define the generalized aggregation operations of those referred to [9] over the given matrix A as follows:

$(\circ) - \alpha_{(K,K_i)}$**-aggregation** – it is aggregation of an index K_i of dimension K

$$\alpha_{(K,K_i,\circ)}(A, K_{i,0})$$

$$
=\left\{\begin{array}{c|ccccc}
H_g \in H & L_1 & \cdots & L_j & \cdots & L_n \\ \hline
K_1 & a_{K_1,L_1,H_g} & \cdots & a_{K_1,L_j,H_g} & \cdots & a_{K_1,L_n,H_g} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
K_{i,0} & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_1,H_g} & \cdots & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_j,H_g} & \cdots & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_n,H_g} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
K_m & a_{K_m,L_1,H_g} & \cdots & a_{K_m,L_j,H_g} & \cdots & a_{K_m,L_n,H_g}
\end{array}\right\}
$$

where $K_i \subset K$ and $1 \le i \le m$

$(\circ) - \alpha_{(K,K_*)}$**-aggregation**

Let be given an index set $K_* \subseteq K$, $K_* = \{K_{v_1}, \dots, K_{v_x}, \dots, K_{v_t}\}, 1 \le v_x \le m$ for $1 \le x \le t$. Let be given $V_* = \{K_{v_1,0}, \dots, K_{v_x,0}, \dots, K_{v_t,0}\}, K_{v_x,0} \notin K$ for $1 \le x \le t$. In this case we will define:

$$
\alpha_{(K,K_*,\circ)}(A, V_*) = \alpha_{(K,K_*,\circ)}(A, \langle K_{v_{1,0}}, \dots, K_{v_{x,0}}, \dots, K_{v_{p,0}} \rangle)
$$

$$
= \alpha_{(K,K_{V_t},\circ)}((\dots \alpha_{(K,K_{v_1},\circ)}(A, K_{v_{1,0}}) \dots), K_{v_{t,0}})
$$

$$
=\left\{\begin{array}{c|cccc}
H_g & L_1 & \cdots & L_j & \cdots \\ \hline
K_1 & a_{K_1,L_1,H_g} & \cdots & a_{K_1,L_j,H_g} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
K_{v_{1,0}} & \underset{k_{v_1,e}\,\in\,K_{v_1}}{\circ} a_{k_{v_1,e},L_1,H_g} & \cdots & \underset{k_{v_1,e}\,\in\,K_{v_1}}{\circ} a_{k_{v_1,e},L_j,H_g} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
K_{v_{t,0}} & \underset{k_{v_t,e}\,\in\,K_{v_p}}{\circ} a_{k_{v_t,e},L_1,H_g} & \cdots & \underset{k_{v_t,e}\,\in\,K_{v_t}}{\circ} a_{k_{v_t,e},L_j,H_g} & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
K_m & a_{K_m,L_1,H_g} & \cdots & a_{K_m,L_j,H_g} & \cdots
\end{array}\right.
$$

$$\left.\begin{array}{cc} \cdots & L_n \\ \hline \cdots & a_{K_1,L_n,H_g} \\ \ddots & \vdots \\ \cdots & \underset{k_{v_1,e}\,\in\,K_{v_1}}{\circ} a_{k_{v_1,e},L_n,H_g} \\ \ddots & \vdots \\ \cdots & \underset{k_{v_t,e}\,\in\,K_{v_t}}{\circ} a_{k_{v_t,e},L_n,H_g} \\ \ddots & \vdots \\ \cdots & a_{K_m,L_n,H_g} \end{array} |H_g \in H \right\}$$

Using operation "Transposition", defined in [3] and extended in [10], similarly can be defined and generalized operations:

$\{(\circ)-\alpha_{(L,L_j)}\}, \{(\circ)-\alpha_{(L,L_*)}\}, \{(\circ)-\alpha_{(H,H_g)}\}, \{(\circ)-\alpha_{(H,H_*)}\}$-aggregation.

Let us expand the definitions for other aggregation operations from [9] and define new aggregation operations:

$\alpha_{(\langle K,K_i\rangle,\langle L,L_j\rangle,\circ)}, \alpha_{(\langle K,K_i\rangle,\langle H,H_g\rangle,\circ)}$, and $\alpha_{(\langle L,L_j\rangle,\langle H,H_g\rangle,\circ)}$ as follows:

$(\circ)-\alpha_{(\langle K,K_i\rangle,\langle L,L_j\rangle)}$**-aggregation**

$$\alpha_{(\langle K,K_i\rangle,\langle L,L_j\rangle,\circ)}(A,\langle K_{i,0},L_{j,0}\rangle)$$

$$= \left\{ \begin{array}{c|ccccc} H_g \in H & L_1 & \cdots & L_{j,0} & \cdots & L_n \\ \hline K_1 & a_{K_1,L_1,H_g} & \cdots & \underset{j,1\le j\le j,J}{\circ} a_{K_1,l_j,H_g} & \cdots & a_{K_1,L_n,H_g} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ K_{i,0} & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_1,H_g} & \cdots & \underset{\substack{i,1\le i\le i,I \\ j_1\le j\le j,J}}{\circ} a_{k_i,l_j,H_g} & \cdots & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_n,H_g} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ K_m & a_{K_m,L_1,H_g} & \cdots & a_{K_m,L_j,H_g} & \cdots & a_{K_m,L_n,H_g} \end{array} \right\}$$

where $K_i \subset K (1 \le i \le m)$ and $L_j \subset L (1 \le j \le n)$.

Let be given an index sets $K_* \subseteq K$ and $K_* = \{K_{v_1}, \ldots, K_{v_x}, \ldots, K_{v_t}\}, 1 \le v_x \le m$ for $1 \le x \le t$ and $L_* \subseteq L$ and $L_* = \{L_{u_1}, \ldots, L_{u_y}, \ldots, L_{u_s}\}, 1 \le u_y \le n$ for $1 \le$

$y \leq s$. Let be given $V_* = \{K_{v_1,0}, \ldots, K_{v_x,0}, \ldots, K_{v_t,0}\}, K_{v_x,0} \notin K$ for $1 \leq x \leq t$; $W_* = \{L_{u_{1,0}}, \ldots, L_{u_{y,0}}, \ldots, L_{u_{s,0}}\}, L_{u_y,0} \notin L$ for $1 \leq y \leq s$.

In this case we will define: $(\circ) - \alpha_{(\langle K,K_* \rangle, \langle L,L_* \rangle)}$**-aggregation**

$$\begin{aligned} &\alpha_{(\langle K,K_* \rangle, \langle L,L_* \rangle, \circ)}(A, V_*, W_*) \\ &= \alpha_{(\langle K,K_* \rangle, \langle L,L_* \rangle, \circ)}(A, \langle K_{v_{1,0}}, \ldots, K_{v_{x,0}}, \ldots, K_{v_{t,0}} \rangle, \langle L_{u_{1,0}}, \ldots, L_{u_{y,0}}, \ldots, L_{u_{s,0}} \rangle) \\ &= \alpha_{(\langle K,K_{v_p} \rangle, \langle L,L_{u_s} \rangle, \circ)}((\ldots \alpha_{(\langle K,K_{v_1} \rangle, \langle L,L_{u_1} \rangle, \circ)}(A, \langle K_{v_{1,0}}, L_{u_{1,0}} \rangle) \ldots), \langle K_{v_{t,0}}, L_{u_{s,0}} \rangle). \end{aligned}$$

Using operation "Transposition", defined in [3] and extended in [10], similarly can be defined operations: $\{(\circ) - \alpha_{(\langle K,K_i \rangle, \langle H,H_g \rangle)}\}$, $\{(\circ) - \alpha_{(\langle L,L_j \rangle, \langle H,H_g \rangle)}\}$-aggregation and they can be generalized.

3.2 Generalized aggregation operations of $3D$-multilayer extended index matrix $\{3D - MLEIM\}$

Let $K_{i,0}^{(p)} \notin K$, $L_{j,0}^{(q)} \notin L$ and $H_{g,0}^{(r)} \notin H$.
The generalized aggregation operation $(\circ) - \alpha_{(K,K_i^{(P)},\ \textbf{p-layer})}$**-aggregation**, which performs aggregation on the p-th layer of dimension K of the matrix A, which is $3D - MLEIM$, has the form:

$$\alpha_{(K,K_i^{(P)},\ \text{p-layer}\ ,\circ)}(A, K_{i,0}^{(p)})$$

$$= \left\{ \begin{array}{c|cc} H_g^{(R)} \in H & L_1^{(Q)} & \cdots \\ \hline K_1^{(P)} & a_{K_1^{(P)}, L_1^{(Q)}, H_g^{(R)}} & \cdots \\ \vdots & \vdots & \ddots \\ K_i^P & a_{K_i^{(P)}, L_1^{(Q)}, H_g^{(R)}} & \cdots \\ \{K_i^P,\ \text{p-layer}\ \} d K_{i,0}^{(p)} & \underset{\substack{1 \leq \rho \leq p-1 \\ K_u^{(\rho)} \in K_{i*}^{(p)}}}{\circ} a_{K_u^{(\rho)}, L_1^{(Q)}, H_g^{(R)}} & \cdots \\ \vdots & \vdots & \ddots \\ K_m^{(P)} & a_{K_m^{(P)}, L_1^{(Q)}, H_g^{(R)}} & \cdots \end{array} \right.$$

$$
\left.\begin{array}{cccc}
\dots & L_j^{(Q)} & \dots & L_n^{(Q)} \\
\hline
\dots & a_{K_1^{(P)},L_j^{(Q)},H_g^{(R)}} & \dots & a_{K_1^{(P)},L_n^{(Q)},H_g^{(R)}} \\
\ddots & \vdots & \ddots & \vdots \\
\dots & \underset{\substack{1 \le \rho \le p-1 \\ K_u^{(\rho)} \in K_{i*}^{(p)}}}{\circ} a_{K_u^{(\rho)},L_j^{(Q)},H_g^{(R)}} & \dots & \underset{\substack{1 \le \rho \le p-1 \\ K_u^{(\rho)} \in K_{i*}^{(p)}}}{\circ} a_{K_u^{(\rho)},L_n^{(Q)},H_g^{(R)}} \\
\ddots & \vdots & \ddots & \vdots \\
\dots & a_{K_m{}^{(P)},L_j^{(Q)},H_g^{(R)}} & \dots & a_{K_m^{(P)},L_n^{(Q)},H_g^{(R)}}
\end{array}\right\}
$$

where $K_i^P \subset K$ and $1 \le p \le P$.

Let be given an index set $K_* \subseteq K, K_* = \{K_{v_1}^{(P)}, \dots, K_{v_x}^{(P)}, \dots, K_{v_t}^{(P)}\}$,
$V_* = \{K_{v_1,0}^{(p)}, \dots, K_{v_x,0}^{(p)}, \dots, K_{v_t,0}^{(p)}\} \notin K$ for $1 \le p \le P$.

In this case we will define $(\circ) - \alpha_{(K,K_*,\text{ \textbf{p-layer}})}$**-aggregation** as follows:

$$\alpha_{(K,K_*,\text{ p-layer},\circ)}(A, V_*) = \alpha_{(K,K_*,\text{ p-layer},\circ)}(A, \langle K_{v_1,0}^{(p)}, \dots, K_{v_x,0}^{(p)}, \dots, K_{v_t,0}^{(p)} \rangle)$$

$$= \alpha_{(K,K_{v_t}^{(P)},\circ)}((\dots \alpha_{(K,K_{v_1}^{(P)},\circ)}(A, K_{v_1,0}^{(p)}) \dots), K_{v_t,0}^{(p)}).$$

Let us expand the defined final aggregation operation. Let be given an index set $K_* \subseteq K$ and $K_* = \{K_{v_1}^{(P)}, \dots, K_{v_x}^{(P)}, \dots, K_{v_t}^{(P)}\}$, $V_* = \{K_{v_1,0}^{(p_1)}, \dots, K_{v_x,0}^{(p_x)}, \dots, K_{v_t,0}^{(p_t)}\} \notin K$ and $P_* = \{p_1, \dots, p_x, \dots, p_t\}$, where $1 \le p_x \le P$ for $1 \le x \le t$. We denote the dimension of G by $dim(G) = u$. Let $dim(K_*) = dim(P_*) = dim(V_*) = t$.

In this case we will define $(\circ) - \alpha_{(K,K_*,P_*)}$**-aggregation** as follows:

$$\alpha_{(K,K_*,P_*,\circ)}(A, V_*) = \alpha_{(K,K_*,P_*,\circ)}(A, \langle K_{v_1,0}^{(p_1)}, \dots, K_{v_x,0}^{(p_x)}, \dots, K_{v_t,0}^{(p_t)} \rangle)$$

$$= \alpha_{(K,K_{v_t}^{(P)},\circ)}((\dots \alpha_{(K,K_{v_1}^{(P)},\circ)}(A, K_{v_1,0}^{(p_1)}) \dots), K_{v_t,0}^{(p_t)}).$$

Using operation "Transposition", defined in [3], similarly can be defined and generalized the operations: $\{(\circ) - \alpha_{(L,L_j^{(Q)},\text{ q-layer})}, (\circ) - \alpha_{(H,H_g^{(R)},\text{ r-layer})}\}$-aggregation.

Let us expand the definitions for other aggregating operations from [9] and let us define: $\alpha_{(\langle K,K_i^{(P)},\text{ p-layer } \rangle,\langle L,L_j^{(Q)},\text{ q-layer } \rangle,\circ)}$, $\alpha_{(\langle K,K_i^{(P)},\text{ p-layer } \rangle,\langle H,H_g^{(R)},\text{ r-layer } \rangle,\circ)}$

and $\alpha_{(\langle L,L_j^{(Q)},\text{ q-layer } \rangle,\langle H,H_g^{(R)},\text{ r-layer } \rangle,\circ)}$-aggregation.

$(\circ) - \alpha_{(\langle K,K_i^{(P)},\text{ \textbf{p-layer} } \rangle,\langle L,L_j^{(Q)},\text{ \textbf{q-layer} } \rangle,\circ)}$**-aggregation**

$$\alpha_{(\langle K,K_i^{(P)},\text{ p-layer } \rangle,\langle L,L_j^{(Q)},\text{ q-layer } \rangle,\circ)}(A, \langle K_{i,0}^{(p)}, L_{j,0}^{(q)} \rangle)$$

$$
= \left\{ \begin{array}{c|ccccc}
 & L_1^{(Q)} & \ldots & L_{j,0}^{(q)} & \ldots & L_n^{(Q)} \\
\hline
K_1^{(P)} & a_{K_1^{(P)},L_1^{(Q)},H_g^{(R)}} & \ldots & \underset{\substack{1 \le \sigma \le q-1 \\ L_v^{\sigma} \in L_{j*}^{(q)}}}{\circ} a_{K_1^{(P)},L_j^{(\sigma)},H_g^{(R)}} & \ldots & a_{K_1^{(P)},L_n^{(Q)},H_g^{(R)}} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
K_{i,0}^{(p)} & \underset{\substack{1 \le \rho \le p-1 \\ K_u^{\rho} \in K_{i*}^{(p)}}}{\circ} a_{K_u^{(\rho)},L_1^{(Q)},H_g^{(R)}} & \ldots & \underset{\substack{1 \le \rho \le P-1 \\ K_u^{\rho} \in K_i^{(p)} \\ 1 \le \sigma \le Q-1 \\ L_v^{\sigma} \in L_j^{(q)}}}{\circ} a_{K_u^{(\rho)},L_j^{(\sigma)},H_g^{(R)}} & \ldots & \underset{\substack{1 \le \rho \le P-1 \\ K_u^{\rho} \in K_{i*}^{(p)}}}{\circ} a_{K_u^{(\rho)},L_n^{(Q)},H_g^{(R)}} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
K_m^{(P)} & a_{K_m^{(P)},L_1^{(Q)},H_g^{(R)}} & \ldots & \underset{\substack{1 \le \sigma \le Q-1 \\ L_v^{\sigma} \in L_j^{(q)}}}{\circ} a_{K_m^{(P)},L_j^{(\sigma)},H_g^{(R)}} & \ldots & a_{K_m^{(P)},L_n^{(Q)},H_g^{(R)}}
\end{array} | H_g^{(R)} \in H \right\}
$$

where $K_i^{(P)} \subset K$ for $1 \le p \le P$, $L_j^{(Q)} \subset L$ for $1 \le q \le Q$.

Using operation "Transposition", defined in [3], similarly can be defined and generalized the operations: $(\circ) - \alpha_{(\langle K, K_i^{(P)}, \text{ p-layer } \rangle, \langle H, H_g^{(R)}, \text{ r-layer } \rangle)}$,

$(\circ) - \alpha_{(\langle H, H_g^{(R)}, \text{ r-layer } \rangle, \langle L, L_j^{(Q)}, \text{ q-layer } \rangle)}$.

4 Properties of aggregation operations

Let A is $3D - EIM$, then can be formulated following:

Theorem 1.

1. $\alpha_{(K,\circ)}(pr_{K_i,L,H}A, K_{i,0}) = pr_{k_{i,0},L,H}(\alpha_{(K,K_i,\circ)}(A, K_{i,0}))$
2. $\alpha_{(L,\circ)}(pr_{K,L_j,H}A, L_{j,0}) = pr_{K,L_{j,0},H}(\alpha_{(L,L_j,\circ)}(A, L_{j,0}))$
3. $\alpha_{(H,\circ)}(pr_{K,L,H_g}A, H_{g,0}) = pr_{K,L,H_{g,0}}(\alpha_{(H,H_g,\circ)}(A, H_{g,0}))$
4. $\alpha_{(K,L,\circ)}(pr_{K_i,L_j,H}A, \langle K_{i,0}, L_{j,0}\rangle) = pr_{K_{i,0},L_{j,0},H}(\alpha_{(\langle K,K_i\rangle,\langle L,L_j\rangle,\circ)}(A, \langle K_{i,0}, L_{j,0}\rangle))$
5. $\alpha_{(K,H,\circ)}(pr_{K_i,L,H_g}A, \langle K_{i,0}, H_{g,0}\rangle) = pr_{K_{i,0},L,H_{g,0}}(\alpha_{(\langle K,K_i\rangle,\langle H,H_g\rangle,\circ)}(A, \langle K_{i,0}, H_{g,0}\rangle))$
6. $\alpha_{(L,H,\circ)}(pr_{K,L_j,H_g}A, \langle L_{j,0}, H_{g,0}\rangle) = pr_{K,L_{j,0},H_{g,0}}(\alpha_{(\langle L,L_j\rangle,\langle H,H_g\rangle,\circ)}(A, \langle L_{j,0}, H_{g,0}\rangle))$.

Proof: Following definition for operation "Projection", given in [10], we can write that, $pr_{K_i,L,H}A = [K_i, L, H, b_{k_i,l_j,h_g}]$, where

$$(\forall k_i \in K_i)(\forall l_j \in L)(\forall h_g \in H)(b_{k_i,l_j,h_g} = a_{k_i,l_j,h_g}).$$

Therefore,

$$pr_{K_i,L,H}A$$

$$= \left\{ \begin{array}{c|ccccccc} H_g \in H & L_1 & \cdots & l_{j,1} & \cdots & l_{j,J} & \cdots & L_n \\ \hline k_{i_1} & a_{k_{i,1},L_1,H_g} & \cdots & a_{k_{i,1},l_{j,1},H_g} & \cdots & a_{k_{i,1},l_{j,J},H_g} & \cdots & a_{k_{i,1},L_n,H_g} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{i,I} & a_{k_{i,I},L_1,H_g} & \cdots & a_{k_{i,I},l_{j,1},H_g} & \cdots & a_{k_{i,I},l_{j,J},H_g} & \cdots & a_{k_{i,I},L_n,H_g} \end{array} \right\}.$$

$$\alpha_{(K,\circ)}(pr_{K_i,L,H}A, K_{i,0})$$

$$= \left\{ \begin{array}{c|ccccc} H_g \in H & L_1 & \cdots & L_j & \cdots & L_n \\ \hline K_{i,0} & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_1,H_g} & \cdots & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_j,H_g} & \cdots & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_n,H_g} \end{array} \right\}$$

From definition

$$\alpha_{(K,i,\circ)}(A, K_{i,0})$$

$$= \left\{ \begin{array}{c|ccccc} H_g \in H & L_1 & \cdots & L_j & \cdots & L_n \\ \hline K_1 & a_{K_1,L_1,H_g} & \cdots & a_{K_1,L_j,H_g} & \cdots & a_{K_1,L_n,H_g} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ K_{i,0} & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_1,H_g} & \cdots & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_j,H_g} & \cdots & \underset{i,1\le i\le i,I}{\circ} a_{k_i,L_n,H_g} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ K_m & a_{K_m,L_1,H_g} & \cdots & a_{K_m,L_j,H_g} & \cdots & a_{K_m,L_n,H_g} \end{array} \right\}$$

Therefore,

$$pr_{K_{i,0},L,H}(\alpha_{(K,i,\circ)}(A, K_{i,0}))$$

$$= \left\{ \begin{array}{c|ccccc} H_g \in H & L_1 & \cdots & L_j & \cdots & L_n \\ \hline & & & & & \\ K_{i,0} & \underset{i,1 \le i \le i,I}{\circ} a_{k_i,L_1,H_g} & \cdots & \underset{i,1 \le i \le i,I}{\circ} a_{k_i,L_j,H_g} & \cdots & \underset{i,1 \le i \le i,I}{\circ} a_{k_i,L_n,H_g} \end{array} \right\}$$

Therefore, the Proposition 1 is proven. The proof of statements from 2. to 6. is similarly.

5 Conclusion

The defined MLEIM is a tool for presenting of an "Olap-cube" [8], which provides interactive query-driven analysis of accumulated business data for the purpose of decision making. By this reason, in a next research we will discuss the application of the MLEIMs and the defined aggregation operations for a description of some operations of an "Olap-cube".

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