A NOTE ON NEW PARTIAL ORDERING
OVER INTUITIONISTIC FUZZY SETS

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Abstract: A new partial ordering between intuitionistic fuzzy sets is proposed and investigated in the present paper. Some results concerning its relation to the classical ordering are given.

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1 Introduction

Intuitionistic fuzzy sets (IFS) were introduced by K. Atanassov (see [1]) as a generalization and extension of the concept of fuzzy sets. We will briefly remind some of the basic definitions and notions.

Let $X$ be a universe set, $A \subseteq X$, $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ are mappings reflecting the degree of membership and non-membership of the element $x \in X$ to the set $A$, respectively, such that for every $x$ it is fulfilled that

$$\mu_A(x) + \nu_A(x) \leq 1 \quad (1)$$

Definition 1. Following [2], we call the set

$$A^* \overset{\text{def}}{=} \{x, \mu_A(x), \nu_A(x) | x \in E\}$$

an intuitionistic fuzzy set (IFS) and the mapping $\pi_A : X \to [0, 1]$, which is given by

$$\pi_A(x) \overset{\text{def}}{=} 1 - \mu_A(x) - \nu_A(x), \quad (2)$$

is called intuitionistic fuzzy index (sometimes also: hesitancy margin or degree of indeterminacy) of the element $x$ (cf. [6]).
Further we denote the class of all IFSs defined over a universe set \(X\) by \(\text{IFS}(X)\).

**Definition 2** (cf. [2, p.134, (7.1)], [6, p.43, Definition 3.4]). For a given IFS \(A \in \text{IFS}(X)\) the degree of definiteness of the element \(x\) is said to be:

\[
\sigma_{1,A}(x) \overset{\text{def}}{=} \mu_A(x) + \nu_A(x)
\]

This degree provides an intuitive measure of the certainty of the knowledge established for the element. Indeed it is easy to see that it is directly related to intuitionistic fuzzy index, since for all \(x \in X\), we have (due to (2)):

\[
\sigma_{1,A}(x) = 1 - \pi_A(x).
\]

**Definition 3** (cf. [3]). An intuitionistic fuzzy pair (IFP) is an ordered couple of real non-negative numbers \((a_1, a_2)\), with the constraint:

\[
a_1 + a_2 \leq 1.
\]

This concept is very important in practice since many methods implementing IFSs, lead to estimates in the form of IFPs as is the case of InterCriteria Analysis [4]. One way to measure which result is “better” is by using some partial ordering on IFPs. The one most often used is the following:

**Definition 4** (cf. [3]). Given two IFPs \(u = (u_1, u_2)\) and \(v = (v_1, v_2)\), we say that \(u\) is less or equal to \(v\), and we write:

\[
u \leq v,
\]

iff

\[
\begin{align*}
  u_1 &\leq v_1 \\
  u_2 &\geq v_2.
\end{align*}
\]

**Remark 1.** It is obvious that the above is a partial ordering, since it is transitive, reflexive and antisymmetric but there exist \(u\) and \(v\), for which conditions (5) are not satisfied.

**Remark 2.** An equivalent form of (5) is:

\[
\begin{align*}
  u_1 &\leq v_1 \\
  1 - u_2 &\leq 1 - v_2.
\end{align*}
\]

A formulation of the necessary and sufficient condition for the fulfillment of (5) is made obvious by (6) (due to (4)), namely:

\[
\begin{align*}
  \min(u_1, v_1, 1 - u_2, 1 - v_2) &= u_1 \\
  \max(u_1, v_1, 1 - u_2, 1 - v_2) &= 1 - v_2.
\end{align*}
\]

Another partial ordering between IFPs investigated by E. Marinov is the following:
Definition 5 (cf. [7]). Given two IFPs \( u = \langle u_1, u_2 \rangle \) and \( v = \langle v_1, v_2 \rangle \), we say that \( u \) is (\( \pi \)-based) less or equal to \( v \), and we write:

\[ u \preceq_\pi v, \]

iff

\[
\begin{aligned}
& u_1 \leq v_1 \\
& u_2 \leq v_2.
\end{aligned}
\] (7)

Remark 3. We note that for any two IFPs \( u \) and \( v \), at least one of the four is satisfied:

- \( u \leq v \)
- \( v \leq u \)
- \( u \preceq_\pi v \)
- \( v \preceq_\pi u \).

This follows directly from the definitions but it is also evident that the two partial orderings are mutually exclusive except in the case of equalities.

Remark 4. The partial ordering \( \preceq \) over IFPs, defines inclusion over IFSs in the following manner. Let \( A, B \in \text{IFS}(X) \). Then we say that \( A \) is included in \( B \), and write \( A \subseteq B \) if for all \( x \in X \):

\[
\langle \mu_A(x), \nu_A(x) \rangle \leq \langle \mu_B(x), \nu_B(x) \rangle.
\] (8)

2 A new partial ordering between IFPs

In what follows we propose a new partial ordering between IFPs, and investigate its relation to the classical \( \preceq \) ordering from Definition 4. Recalling (3), let us first introduce

Definition 6. For a given IFP \( a = \langle a_1, a_2 \rangle \) we will say that the degree of definiteness of \( a \) is:

\[
\sigma(a) \overset{\text{def}}{=} a_1 + a_2
\] (9)

Now we are ready to define the new partial ordering as:

Definition 7. Given two IFPs \( u = \langle u_1, u_2 \rangle \) and \( v = \langle v_1, v_2 \rangle \), we say that \( u \) is (definiteness-based) less or equal to \( v \), and we write:

\[ u \preceq_\sigma v, \]

iff

\[
\begin{aligned}
& \sigma(u_1) \leq \sigma(v_1) \\
& \sigma(u_2) \geq \sigma(v_2).
\end{aligned}
\] (10)

The fact that \( \preceq_\sigma \) is partial ordering (i.e. is transitive, reflexive and antisymmetric) is obvious. We will now establish some results concerning its relation with the \( \preceq \) ordering from Definition 4.
**Theorem 1.** If for two IFPs $u$ and $v$, it is true that $u \preceq v$, then it is also true that $u \leq v$.

**Proof.** Let us consider the only three possible cases:

i. $\sigma(u) = \sigma(v)$

ii. $\sigma(u) > \sigma(v)$

iii. $\sigma(u) < \sigma(v)$

If we have case i., then (10) is equivalent to (5). Hence, the assertion is correct.

Let us consider case ii. Then the first inequality of (10) yields:

$$u_1 \sigma(v) \leq u_2 \sigma(u) \leq v_1 \sigma(v),$$

and hence we establish that $u_1 \leq v_1$. It remains to show that $u_2 \geq v_2$. But

$$\sigma(u) = u_1 + u_2 > v_2 + v_1 = \sigma(v),$$

and it follows that

$$u_2 > v_2 + \underbrace{(v_1 - u_1)}_{\geq 0}.$$ 

Hence, $u_2 \geq v_2$, and this case yields again the validity of (5).

Lastly, let us consider case iii. Then the second inequality of (10) yields:

$$\sigma(v) u_2 \geq \sigma(u) u_2 \geq \sigma(v) v_2,$$

and hence we establish that $u_2 \geq v_2$. It remains to show that $u_1 \leq v_1$. But

$$\sigma(u) = u_1 + u_2 < v_2 + v_1 = \sigma(v),$$

and it follows that

$$u_1 < u_1 + \underbrace{(u_2 - v_2)}_{\geq 0} < v_1$$

Hence, $u_1 \leq v_1$, and this case yields again the validity of (5). This completes the proof. □

**Remark 5.** We have established that $\preceq$ implies $\leq$. The converse is generally not true. For instance, let

$$u = (0.3, 0.5), \quad v = (0.42, 0.48)$$

We obviously have $u \leq v$, but $u \not\preceq v$.

**Definition 8.** Let $A, B \in \text{IFS}(X)$. Then we say that $A$ is included definiteness-wise in $B$, and write $A \subseteq \sigma B$ if for all $x \in X$:

$$\langle \mu_A(x), \nu_A(x) \rangle \preceq \langle \mu_B(x), \nu_B(x) \rangle$$

(11)

A direct Corollary from Theorem 1 is the following

**Corollary 1.** Let $A, B \in \text{IFS}(X)$. Then if $A \subseteq \sigma B$, it follows that $A \subseteq B$. 

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3 Conclusion

In the present paper a new partial ordering between IFPs is introduced and its connection to the classical ordering is established. In passing a necessary and sufficient condition for the fulfillment of the classical ordering is given. Also, the proposed ordering introduces a new inclusion between intuitionistic fuzzy sets.

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References


